

# The *quot* functor of a quasi-coherent sheaf

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## Abstract

We build a *scheme* parametrizing isomorphism classes of *coherent* quotients of a quasi-coherent sheaf on a projective scheme. The main tool to achieve the construction is the classical Grassmannian embedding of the *quot* functor of a coherent sheaf [Gr] combined with a result of Deligne [Del], realizing quasi-coherent sheaves as ind-objects in the category of quasi-coherent sheaves of finite presentation. We end our treatment with the discussion of a special case in which we can retain an analog of the Grassmannian embedding.

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# Introduction

**The original construction and our leading question.** Since their introduction in [Gr], *Quot* schemes have played a fundamental role in algebraic geometry and, in particular, in deformation theory. For instance, they provide natural compactifications of spaces of morphisms between certain schemes [Gr], they are used to give a presentation of the stack of coherent sheaves over a projective scheme [LMB], and their derived version [CFK] is of fundamental importance in derived algebraic geometry.

Recall that given a Hilbert polynomial  $h$ , a projective scheme  $X$  over an algebraically closed field  $\mathbf{k}$  and a quasi-coherent sheaf  $\mathcal{E}$ , one defines the contravariant functor  $quot_{\mathcal{E}/X}^h : (\mathbf{Sch}/\mathbf{k})^\circ \rightarrow \mathbf{Set}$  as

$$quot_{\mathcal{E}/X}^h(T) := \left\{ \mathcal{K} \subset \mathcal{E}_T \mid \begin{array}{l} \mathcal{E}_T/\mathcal{K} \text{ is coherent, flat over } \mathcal{O}_T, \\ \text{and has Hilbert polynomial } h \end{array} \right\}$$

where  $\mathcal{E}_T := \mathcal{E} \otimes_{\mathbf{k}} \mathcal{O}_T$ , and as pullback on morphisms. In his original outline of the construction, Grothendieck proves the representability of the above functor only when  $\mathcal{E}$  is *coherent*. Despite its classical nature, the representability of the *quot* functor for a general quasi-coherent  $\mathcal{E}$  has not been addressed. This is exactly the question we answer in the present paper. More precisely, the theorem below follows from our main result (Theorem 2.2.8).

**Theorem 0.0.1.** *Let  $\mathcal{E}$  be a quasi-coherent  $\mathcal{O}_X$ -module and let  $X$  and  $h$  be as above. Then there is a scheme  $\mathrm{IQuot}_h^X(\mathcal{E})$  representing the functor  $quot_{\mathcal{E}/X}^h$ .*

In the following two paragraphs we give an idea of the construction of the scheme  $\mathrm{IQuot}_h^X(\mathcal{E})$  and provide an outline of the other results contained in this paper. In the rest of this introduction  $\mathcal{E}$  will always denote a quasi-coherent sheaf over the projective scheme  $X$ .

**The filtering schematic Grassmannian.** The main idea in Grothendieck's paper is that the representability of the *quot* functor and the resulting universal property of the *Quot* scheme are inherited from the corresponding properties of a certain Grassmannian, in which the *quot* functor lives. Motivated by this, in Section 1 we provide a filtering construction of the Grassmannian. More in detail, if  $\mathrm{Grass}_n(\mathcal{F})$  denotes the Grassmannian of locally free rank  $n$  quotients of a quasi-coherent  $\mathcal{F}$  [EGAS], we prove the following proposition (see Lemma 1.3.5).

**Proposition 0.0.2.** *Let  $\mathcal{F}$  be a quasi-coherent sheaf over a scheme  $S$ . Then  $\text{Grass}_n(\mathcal{F})$  is the filtering inductive limit over  $i$  of an increasing sequence of quasi-compact open subschemes  $\left(\underline{G}(n, \mathcal{F})_i\right)_{i \in I}$ .*

The schemes  $\underline{G}(n, \mathcal{F})_i$  in the statement are constructed as projective limits of diagrams consisting of certain subschemes of Grassmannians of finite type and affine morphisms between them (see Lemma 1.3.3 for the proof of affineness). Also, note that a crucial ingredient in the proof of Proposition 0.0.2 is the following theorem of Deligne.

**Lemma 0.0.3.** *[Del] Let  $X$  be a quasi-compact quasi-separated scheme (not necessarily Noetherian). Then the category  $\mathcal{QCoh}(X)$  of quasi-coherent sheaves on  $X$  is equivalent to that of ind-objects in the category of quasi-coherent sheaves of finite presentation on  $X$ .*

For the sake of the reader, we briefly review the concepts involved the above statement in Subsection 1.1.

**Two uses of the Grassmannian embedding.** The filtering construction from Section 1 will be of twofold interest to us. First, in building the  $Quot$  scheme of  $\mathcal{E}$  we will proceed in a way that follows the same “ind-pro principle” of Section 1, to get a scheme which we call  $\text{IQuot}_h^X(\mathcal{E})$  to stress its infinite dimensional nature. More precisely, we construct a candidate for the scheme representing the functor  $quot_{\mathcal{E}/X}^h$  as a filtering inductive limit of certain schemes denoted  $\underline{Q}(h, \mathcal{E})_i$ . In order to obtain the  $\underline{Q}(h, \mathcal{E})_i$ ’s, we take the projective limit of a filtering projective system consisting of some open subschemes of ordinary  $Quot$  schemes and affine morphisms between them. Roughly speaking, the affineness of such morphisms will be proved by viewing them as restrictions of morphisms between Grassmannians (Lemma 2.2.1).

In the last part we introduce *uniformly regular* sheaves over a projective scheme  $X$ . These are quasi-coherent sheaves all of whose coherent approximations in the sense of Proposition 1.1.1 (and the following Remarks) have Castelnuovo-Mumford regularities [Mum] bounded by a given integer  $m$ . This said, the other way in which we use the filtering construction from Proposition 0.0.2 is to show that  $\text{IQuot}_h^X(\mathcal{E})$  can be embedded in some schematic Grassmannian. The precise statement is as follows.

**Theorem 0.0.4.** *Let  $\mathcal{E}$  be a uniformly  $m$ -regular quasi-coherent sheaf on a  $\mathbf{k}$ -projective scheme  $X$ . Then there is a quasi-closed embedding*

$$\mathrm{IQuot}_h^X(\mathcal{E}) \hookrightarrow \mathrm{Grass}_{h(m)} \left( \varinjlim_j H^0(X, \mathcal{E}^j(m)) \right).$$

By a quasi-closed embedding, we understand the inductive limit of a ladder diagram whose rungs are closed embeddings of schemes (Definition 2.3.4).

**Further directions of research.** Assume  $\mathbf{k} = \mathbb{C}$ . Recall that given nonnegative integers  $d, r$  and  $m$ , with  $r < m$ , the ordinary *Quot* scheme  $\mathrm{Quot}_{r,d}^{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1}^m)$  of quotients of  $\mathcal{O}_{\mathbb{P}^1}^m$  of rank  $r$  and degree  $d$ , can be used to compactify the space

$$\mathcal{M}_d^m := \mathrm{Rat}_d(\mathbb{P}^1, \mathrm{Grass}_r(\mathcal{O}_{\mathbb{P}^1}^m)),$$

of maps of degree  $d$  from  $\mathbb{P}^1$  to  $\mathrm{Grass}_r(\mathcal{O}_{\mathbb{P}^1}^m)$ . To see this, one can note that giving a morphism  $\mathbb{P}^1 \rightarrow \mathrm{Grass}_r(\mathcal{O}_{\mathbb{P}^1}^m)$  of degree  $d$  is equivalent to the datum of a quotient bundle of  $\mathcal{O}_{\mathbb{P}^1}^m$  of rank  $r$  and degree  $d$ . Similarly, letting  $m$  tend to  $\infty$ , one could use the *IQuot* scheme we construct here to study the topology of the spaces

$$\mathcal{M}_d^\infty = \mathrm{Rat}_d(\mathbb{P}^1, \mathrm{Grass}_r(\mathcal{O}_{\mathbb{P}^1}^{\oplus \infty})),$$

of rational curves of degree  $d$  in the schematic Grassmannian.

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# 1 A filtering cover for the schematic Grassmannian

## 1.1 Ind-objects and a theorem of Deligne

Recall that a *filtering inductive limit* is defined as the inductive limit of a functor  $F : \mathcal{I} \rightarrow \mathcal{C}$  where  $\mathcal{C}$  is any category and  $\mathcal{I}$  is a filtering poset. In case

no confusion is possible, we may refer to  $\mathcal{I}$  simply as the indexing category. An *ind-object* in the category  $\mathcal{C}$  is by definition a filtering inductive limit in  $\mathcal{C}^* := \mathcal{F}un(\mathcal{C}^\circ, \mathbf{Set})$  of functors representing objects of  $\mathcal{C}$ . One denotes by  $\mathrm{Ind}(\mathcal{C})$  the full subcategory of  $\mathcal{C}^*$  whose objects are the ind-objects of  $\mathcal{C}$  (see [Del] or [MacL] for further details).

The theorem below will be of crucial importance in what follows.

**Proposition 1.1.1.** *[Del, Prop. 2] Let  $X$  be a quasi-compact quasi-separated scheme (not necessarily Noetherian). Then the category  $\mathcal{QCoh}(X)$  of quasi-coherent sheaves on  $X$  is equivalent to that of ind-objects in the category of quasi-coherent sheaves of finite presentation on  $X$ .*

**Remarks 1.1.2.** (1) A quasi-coherent sheaf  $\mathcal{E}$  on a quasi-compact quasi-separated scheme  $X$  is therefore given by the inductive limit in  $\mathcal{QCoh}(X)$  of a filtering inductive diagram

$$(1.1.3) \quad (\mathcal{E}^i, \alpha^{i,j} : \mathcal{E}^i \rightarrow \mathcal{E}^j)_{i,j \in \mathcal{I}}$$

of morphisms of finitely-presented sheaves.

(2) Assuming that  $X$  is Noetherian, the theorem implies that there is an equivalence of categories between  $\mathcal{QCoh}(X)$  and  $\mathrm{Ind}(\mathrm{Coh}(X))$ , the ind-category of the category of coherent sheaves on  $X$ .

## 1.2 Reminder on the classical construction

We start by recalling the construction of the Grassmannian. For an integer  $n \geq 1$ , a scheme  $S$  and quasi-coherent  $\mathcal{O}_S$ -module  $\mathcal{E}$ , we denote by  $\mathcal{Grass}_n(\mathcal{E})$  the set of locally free rank  $n$  quotient  $\mathcal{O}_S$ -modules of  $\mathcal{E}$ .

**Theorem 1.2.1** ([EGAS]). *For every scheme  $S$  and every quasi-coherent  $\mathcal{O}_S$ -module  $\mathcal{E}$ , the functor  $\gamma_{\mathcal{E}}^n : (\mathbf{Sch}/S)^\circ \rightarrow \mathbf{Set}$  given by*

$$\gamma_{\mathcal{E}}^n(T) := \mathrm{grass}_n(\mathcal{E}_T),$$

*where  $\mathcal{E}_T$  is the base change along the structure morphism  $T \rightarrow S$ , is represented by a separated  $S$ -scheme denoted by  $\mathrm{Grass}_n(\mathcal{E})$ . Moreover, there exists a locally free rank  $n$  quotient  $\mathcal{O}_{\mathrm{Grass}_n(\mathcal{E})}$ -module of  $\mathcal{E}_{\mathrm{Grass}_n(\mathcal{E})}$  denoted  $\mathcal{Q}$  and determined up to a unique isomorphism, such that*

$$g \mapsto g^*(\mathcal{Q}) : \mathrm{Hom}_S(T, X) \xrightarrow{\sim} \gamma_{\mathcal{E}}^n(T)$$

*is a natural isomorphism.*

The vector bundle  $\mathcal{Q}$  in the statement is the *universal quotient bundle* of the Grassmannian. Note that if we do not assume that  $\mathcal{E}$  in Theorem 1.2.1 is of finite type or finitely presented, then the scheme  $\text{Grass}_n(\mathcal{E})$  will not in general be of finite type nor will it be quasi-compact. We will refer to  $\text{Grass}_n(\mathcal{E})$  as the *schematic Grassmannian* when  $\mathcal{E}$  does not have any finiteness properties.

Taking Theorem 1.2.1 for granted when  $\mathcal{E}$  is a *finitely presented* sheaf, we provide a construction of the schematic Grassmannian which is a filtering version of that of [EGAS]. Our construction of the *Quot* scheme of a quasi-coherent sheaf in the second part of this paper will partly follow the same pattern.

In the rest of this Subsection, we briefly review the part of the proof of Theorem 1.2.1 which we will need in the sequel. First, note that insisting on locally free rank  $n$  quotients of  $\mathcal{E}$  in the definition of  $\gamma_{\mathcal{E}}^n$  implies that such a functor is a sheaf of sets ([EGAS]). Therefore, we can reduce to proving its representability over the category of affine schemes. We will make such an assumption until the end of this section.

Let then  $T$  be an  $S$ -scheme. For some index  $i$ , denote by  $\gamma_{\mathcal{E},i}^n(T)$  the subset of  $\gamma_{\mathcal{E}}^n(T)$  consisting of the quotients  $\mathcal{H}$  of  $\mathcal{E}_T$  such that, for some finitely-presented subsheaf  $\mathcal{E}^i$  of  $\mathcal{E}$ , we have a surjective composition

$$(1.2.2) \quad \mathcal{E}_T^i \rightarrow \mathcal{E}_T \twoheadrightarrow \mathcal{H},$$

where the second arrow is the canonical quotient map. Note that the existence of an  $i$  such that a surjective composition as in (1.2.2) exists follows from the fact that the quotient  $\mathcal{H}$  is of finite type and from quasi-compactness of the base scheme. Thus

$$T \mapsto \gamma_{\mathcal{E},i}^n(T),$$

together with the usual pull-back on morphisms, defines a subfunctor of  $\gamma_{\mathcal{E}}^n$ .

To simplify the notation, we will write  $\gamma$  and  $\gamma_i$  instead of  $\gamma_{\mathcal{E}}^n$  and  $\gamma_{\mathcal{E},i}^n$ , respectively.

Since  $S$  is assumed to be affine, the sheaf  $\mathcal{E}$  is completely determined by a  $\Gamma(S, \mathcal{O}_S)$ -module  $E$  via Serre's functor:  $\mathcal{E} = \tilde{E}$ . Thus  $\mathcal{E}$  is generated by a

(possibly infinite) family of sections  $(t_a)_{a \in \Omega}$ . For an  $S$ -scheme  $T$ , denote by  $t_{a,T}$  the pullback of  $t_a$  along the structure morphism  $T \rightarrow S$ . Let then  $H$  be a subset of  $\Omega$  consisting of  $n$  elements, using the sections  $(t_{a,T})_{a \in H}$  we can define a homomorphism of  $\mathcal{O}_T$ -modules

$$\varphi_{H,T} : \mathcal{O}_T^n \rightarrow \mathcal{E}_T.$$

Now, consider the subset  $F_H(T)$  of  $\gamma(T)$  consisting of the quotients  $\mathcal{H}$  of  $\mathcal{E}_T$  such that we have a surjective composition

$$(1.2.3) \quad \mathcal{O}_T^n \xrightarrow{\varphi_{H,T}} \mathcal{E}_T \xrightarrow{q} \mathcal{H},$$

where the second arrow is the canonical quotient map. For future reference we denote by  $s_a$  the canonical image in  $\Gamma(T, \mathcal{H})$  of the section  $t_{a,T}$ .

The datum  $T \mapsto F_H(T)$  together with pullback on morphisms defines a subfunctor of  $\gamma$ , and the main step in Grothendieck's construction consists in proving that such a functor is represented by a scheme  $X_H$  (which one could call the *inverse Plücker subscheme*) which is affine over  $S$ , and that  $F_H$  is an open subfunctor of  $\gamma$ . The functors  $\mathcal{G}_i^j$  in the Lemma 1.3.3 below are essentially unions of the functors  $F_H$  as  $H$  ranges over the set of sections of  $\mathcal{E}^i$ .

### 1.3 The schematic Grassmannian as a filtering inductive limit

The following Lemma along the lines of [EGAS] collects a few results which we will need in the rest of the paper.

**Lemma 1.3.1.** *Let  $Z$  and  $Z'$  be two locally ringed spaces and let  $u : \mathcal{F} \rightarrow \mathcal{G}$  be a homomorphism of quasi-coherent  $\mathcal{O}_Z$ -modules of finite presentation. Then the following statements hold.*

- (a) *The set of points  $z$  of  $Z$  where the localization  $u_z : \mathcal{F}_z \rightarrow \mathcal{G}_z$  is surjective is open in  $Z$ .*
- (b) *The homomorphism  $u_z : \mathcal{F}_z \rightarrow \mathcal{G}_z$  is surjective if and only if the homomorphism  $u_z \otimes 1 : \mathcal{F}_z / \mathfrak{m}_z \mathcal{F}_z \rightarrow \mathcal{G}_z / \mathfrak{m}_z \mathcal{G}_z$  is surjective.*
- (c) *Let  $f : Z' \rightarrow Z$  be a morphism of locally ringed spaces and put  $\mathcal{F}' = f^*(\mathcal{F})$ ,  $\mathcal{G}' = f^*(\mathcal{G})$  and  $u' = f^*(u) : \mathcal{F}' \rightarrow \mathcal{G}'$ . Then the localization  $u'_z$ , at a point  $z'$*

of  $Z'$  is surjective if and only if the localization  $u_z$  is surjective at the point  $z = f(z')$ .

*Proof:* (a) Assume  $u_z$  to be surjective at the point  $z$ . We will find a neighborhood  $N$  of  $z$  such that  $u_{z''}$  is surjective at  $z''$  for all  $z'' \in N$ . For this, since  $\mathcal{F}$  and  $\mathcal{G}$  are of finite presentation, there exists a neighborhood  $U$  of  $z$  such that we have a commutative ladder diagram with exact rows

$$\begin{array}{ccccccc} \mathcal{O}_Z^s|U & \longrightarrow & \mathcal{O}_Z^r|U & \xrightarrow{v} & \mathcal{F} & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow u & & \\ \mathcal{O}_Z^{s_1}|U & \longrightarrow & \mathcal{O}_Z^{r_1}|U & \longrightarrow & \mathcal{G} & \longrightarrow & 0, \end{array}$$

for some positive integers  $r, r_1, s, s_1$ . By exactness of the top row, the surjectivity of  $u$  is equivalent to the surjectivity of the composite  $\varphi := u \circ v : \mathcal{O}_Z^r|U \rightarrow \mathcal{G}$  and the same statement holds for the localized homomorphism  $\varphi_z$ . In order to conclude, we claim that  $\varphi_z$  is surjective if and only if there exists a neighborhood  $V$  of  $z$  such that the localization  $\varphi_{z''}$  is surjective for all  $z''$  in  $V$ .

The question being local, consider instead a ring  $A$ , an  $A$ -module  $N$ , and a homomorphism  $\varphi : A^r \rightarrow N$  such that at some point  $\mathfrak{p}$  the elements  $n_1, \dots, n_r \in N$  generate  $N_{\mathfrak{p}}$  over  $A_{\mathfrak{p}}$ . Moreover, let  $l_1, \dots, l_t$  be generators for  $N$  so that we have

$$(1.3.2) \quad l_i = \sum_{j=1}^r a_{ij} n_j$$

where  $a_{ij} \in A_{\mathfrak{p}}$  for all  $i$  and  $j$ .

Now, the localization of  $A$  at  $\mathfrak{p}$  is an inductive limit

$$A_{\mathfrak{p}} = A[(A \setminus \mathfrak{p})^{-1}] = \varinjlim_{S \not\subset \mathfrak{p}} A[S^{-1}]$$

and, since the  $a_{ij}$  are finite in number, there exists one multiplicatively closed subset  $S$  of  $A$  such that  $a_{ij} \in A[S^{-1}]$  for all  $i, j$ . It follows that we can localize (1.3.2) again at other points around  $\mathfrak{p}$  and get every time a surjective localized homomorphism.

(b) The question is local so consider a homomorphism of modules  $u_{\mathfrak{p}} : M_{\mathfrak{p}} \rightarrow N_{\mathfrak{p}}$  and the canonical quotient morphism  $w_N : N_{\mathfrak{p}} \rightarrow N_{\mathfrak{p}}/\mathfrak{p}N_{\mathfrak{p}}$ . Suppose  $u_{\mathfrak{p}} \otimes 1 : M_{\mathfrak{p}}/\mathfrak{p}M_{\mathfrak{p}} \rightarrow N_{\mathfrak{p}}/\mathfrak{p}N_{\mathfrak{p}}$  is surjective. Then, composing the other canonical



surjection  $w_M : M_{\mathfrak{p}} \rightarrow M_{\mathfrak{p}}/\mathfrak{p}M_{\mathfrak{p}}$  with  $u_{\mathfrak{p}} \otimes 1$  we get a surjective homomorphism  $M_{\mathfrak{p}} \rightarrow N_{\mathfrak{p}}/\mathfrak{p}N_{\mathfrak{p}}$ . By Nakayama's Lemma  $N_{\mathfrak{p}}$  has the same generators of  $N_{\mathfrak{p}}/\mathfrak{p}N_{\mathfrak{p}}$  hence our claim. The other direction of the argument is clear.

(c) We have  $\mathcal{F}'_{z'} = \mathcal{F}_z \otimes_{\mathcal{O}_z} \mathcal{O}_{z'}$  and  $\mathcal{G}'_{z'} = \mathcal{G}_z \otimes_{\mathcal{O}_z} \mathcal{O}_{z'}$  and  $u'_{z'}$  is obtained from  $u_z$  by base change from  $\mathcal{O}_z$  to  $\mathcal{O}_{z'}$ . Let  $\kappa$  and  $\kappa'$  be the residue fields of  $z \in Z$  and  $z' \in Z'$ , then  $\mathcal{F}'_{z'} \otimes \kappa' = (\mathcal{F}_z \otimes \kappa') \otimes \kappa'$  and  $\mathcal{G}'_{z'} \otimes \kappa' = (\mathcal{G}_z \otimes \kappa') \otimes \kappa'$  and we get  $u_{z'} \otimes 1_{\kappa'} : \mathcal{F}'_{z'} \otimes \kappa' \rightarrow \mathcal{G}'_{z'} \otimes \kappa'$  by base changing  $u_z \otimes 1_{\kappa} : \mathcal{F}_z \otimes \kappa \rightarrow \mathcal{G}_z \otimes \kappa$  from  $\kappa$  to  $\kappa'$ . Now, since such a base change is faithfully flat one can conclude by applying Nakayama and part (b).  $\square$

**Lemma 1.3.3.** *Let  $S$  and  $T$  be as in the previous Subsection and let  $\mathcal{E}^i \xrightarrow{\alpha^{i,j}} \mathcal{E}^j$  be a homomorphism of coherent  $\mathcal{O}_S$ -modules. Denote by  $q : \mathcal{E}_T^j \rightarrow \mathcal{H}$  the canonical quotient map. Then the functor*

$$\mathcal{G}_i^j : (\mathbf{Sch}/S)^{\circ} \rightarrow \mathbf{Set}$$

defined by

(1.3.4)

$$\gamma_{\mathcal{E}^j, i}^n(T) = \mathcal{G}_i^j(T) = \left\{ \mathcal{H} \in \gamma_{\mathcal{E}^j}^n(T) \mid \begin{array}{l} \text{the composition } \mathcal{E}_T^i \xrightarrow{\alpha_T^{i,j}} \mathcal{E}_T^j \xrightarrow{q} \mathcal{H} \\ \text{is surjective} \end{array} \right\}$$

is an open subfunctor of  $\gamma_{\mathcal{E}^j}^n$ . Moreover, if we let  $G_i^j$  be the open subscheme of  $\text{Grass}_n(\mathcal{E}^j)$  representing the above subfunctor, we have that there is an affine morphism

$$v_{ij} : G_i^j \rightarrow \text{Grass}_n(\mathcal{E}^i).$$

*Proof:* We start by proving that  $\mathcal{G}_i^j$  is an open subfunctor of  $\gamma_{\mathcal{E}^j}^n$ . Let thus  $Z$  be an  $S$ -scheme, we need to show that the fiber product functor

$$T \mapsto \mathcal{G}_i^j(T) \times_{\gamma_{\mathcal{E}^j}^n(T)} \text{Hom}_S(T, Z)$$

is represented by an open subscheme of  $Z$ . Yoneda's lemma implies that a natural transformation  $\text{Hom}_S(-, Z) \Rightarrow \gamma_{\mathcal{E}^j}^n$  is completely determined by an element  $\mathcal{F} \in \gamma_{\mathcal{E}^j}^n(Z)$  as a pullback:  $\text{Hom}_S(T, Z) \ni g \mapsto g^*(\mathcal{F}) \in \gamma_{\mathcal{E}^j}^n(T)$ . By Lemma 1.3.1 we have that the set of points of  $Z$  where the localization of the composition  $q \circ i_Z$  is surjective is an open subset  $U_{Z, i, \mathcal{F}}$  of  $Z$  (which is equal to the union of the sets  $U_{Z, H, \mathcal{F}}$  of [EGAS] as  $H$  varies in the family of subsets of cardinality  $n$  of  $(t_{a, Z})_{a \in I}$ ). Moreover if  $Y$  is another scheme, the

set of  $S$ -morphisms  $g : Y \rightarrow Z$  such that  $g^*(\mathcal{F}) \in \mathcal{G}_i^j(Y)$  is exactly the inverse image  $g^{-1}(U)$ , again by Lemma 1.3.1. Now, saying that  $g^*(\mathcal{F}) \in \mathcal{G}_i^j(Y)$  means that  $g^{-1}(U)$  must coincide with all of  $Y$ . We have just proved that the above fiber product of functors is represented by an open subscheme of  $Z$ .

To establish the second part of the statement, consider the natural transformation

$$\mathcal{G}_i^j \implies \gamma_{\mathcal{E}^i}^n$$

defined by sending the quotients in  $\mathcal{G}_i^j(T)$  to the corresponding elements of  $\gamma_{\mathcal{E}^i, n}(T)$ . We claim that such morphism of representable functors corresponds to the morphism of schemes  $v_{ij}$  of the statement. To see this, we need to show that the morphism is indeed affine. Now, replacing  $\mathcal{E}$  with  $\mathcal{E}^i$  in (1.2.3), we still obtain subfunctors  $F_H^i$  of  $\gamma_{\mathcal{E}^i}^n$  which are represented by affine subschemes  $X_H^i$ . Since we are assuming  $S$  to be affine, the schemes  $X_H^i$ , as  $H$  varies over the subsets of  $(t_{a,T})_{a \in I}$  of cardinality  $n$ , can be identified to subschemes forming an open covering of  $\text{Grass}_n(\mathcal{E}^i)$ . Given one of such schemes, which we call again  $X_H^i$  by abuse of notation, we have to show that its inverse image is an open affine subscheme of  $G_i^j$ . But, given the homomorphism  $\mathcal{E}^i \xrightarrow{\alpha^{i,j}} \mathcal{E}^j$  over  $S$  and a morphism  $T \rightarrow S$ , the fact that the composition

$$\mathcal{E}_T^i \xrightarrow{\alpha_T^{i,j}} \mathcal{E}_T^j \xrightarrow{q} \mathcal{H}$$

is surjective implies that the inverse image  $v_{ij}^{-1}(X_H^i)$  of  $X_H^i \subset \text{Grass}_n(\mathcal{E}^i)$  is equal to the the subscheme  $X_H^j$  of  $G_i^j$ .  $\square$

**Lemma 1.3.5.** *The functor  $\gamma_i$  is represented by the quasi-compact scheme given by the projective limit*

$$\underline{G}(n, \mathcal{E})_i := \varprojlim_{j > i} G_i^j.$$

*Proof:* First, we show that  $\underline{G}(n, \mathcal{E})_i$  is quasi-compact. For this, recall that the projective limit of a filtering projective system of quasi-compact schemes and affine morphisms exists and is a quasi-compact scheme (see [EGAIV3]). On the other hand, the morphism in the filtering projective system

$$(1.3.6) \quad ((G_i^j)_{j \geq i}, G_i^j \leftarrow G_i^{j'})$$

whose target is  $G_i^i = \text{Grass}_n(\mathcal{E}^i)$ , is affine by Lemma 1.3.3, and all of the others are affine by a similar argument (the morphisms between them can be defined via natural transformations as in the proof of the previous Lemma), whence the quasi-compactness of the projective limit.

To prove that the functor  $\gamma_i$  is representable, we have to show that the functors  $\text{Hom}_S(-, \varprojlim(n, \mathcal{E})_i)$  and  $\gamma_i$  are naturally isomorphic. For this, using

$$\mathcal{G}_i^j \xrightarrow{\sim} \text{Hom}_S(-, G_i^j)$$

and, for any  $S$ -scheme  $T$ ,

$$\text{Hom}_S(T, \varprojlim(n, \mathcal{E})_i) = \text{Hom}_S(T, \varprojlim_{j>i} G_i^j) = \varprojlim_{j>i}^{\mathbf{Set}} \text{Hom}_S(T, G_i^j),$$

one can easily see that  $\gamma_i(T)$  is the vertex of a right cone over the diagram given by the sets  $(\text{Hom}_S(T, G_i^j))_j$  with morphisms resulting from those of the system (1.3.6). Finally, Yoneda's Lemma implies that any natural morphism from  $\text{Hom}_S(-, \varprojlim_{j>i} G_i^j)$  to  $\gamma_i$  is completely determined by pulling back an element of  $\gamma_i(\varprojlim_{j>i} G_i^j)$ , so the square

$$\begin{array}{ccc} \text{Hom}_S(T', \varprojlim_{j>i} G_i^j) & \longrightarrow & \gamma_i(T') \\ \downarrow & & \downarrow \\ \text{Hom}_S(T, \varprojlim_{j>i} G_i^j) & \longrightarrow & \gamma_i(T). \end{array}$$

is commutative thanks to the fact that pulling back anticommutes with the composition of maps.  $\square$

**Lemma 1.3.7.**  *$\gamma_i$  is an open subfunctor of  $\gamma$ .*

*Proof:* The same argument we used in the proof of Lemma 1.3.3 to prove that  $\mathcal{G}_i^j$  is an open subfunctor of  $\gamma_{\mathcal{E}^j}^n$  can be used to show that  $\gamma_i$  is an open subfunctor of  $\gamma := \text{Grass}_n^{\mathcal{E}}$ .  $\square$

**Lemma 1.3.8.** *For  $i < i'$  we have an open embedding of quasi-compact schemes*

$$\varprojlim(n, \mathcal{E})_i \rightarrow \varprojlim(n, \mathcal{E})_{i'}.$$

*Proof:* As we saw in Lemma 1.3.5, for any  $i$  the quasi-compact scheme  $\underline{G}(n, \mathcal{E})_i$  represents the functor  $\gamma_i$ . Therefore, proving the statement amounts to showing that  $\gamma_i$  is an open subfunctor of  $\gamma_{i'}$  whenever  $i < i'$ , namely, that the functor  $(\mathbf{Sch}/S)^\circ \rightarrow \mathbf{Set}$  given by

$$T \mapsto \gamma_i(T) \times_{\gamma_{i'}(T)} \mathrm{Hom}_S(T, Z),$$

is represented by an open subscheme of  $Z$ . As we did in the previous proofs, after applying Yoneda's lemma the main tool we use is the following variation of [EGAS, Lem. 1,9.7.4.6] whose proof can be obtained in essentially the same way.

**Lemma 1.3.9.** (1) *Let  $Z$  be an  $S$ -scheme,  $\mathcal{F}$  a quotient  $\mathcal{O}_Z$ -module of  $\mathcal{E}_Z$  such that the composition*

$$\mathcal{E}_Z^{i'} \xrightarrow{\alpha_Z^{i'}} \mathcal{E}_Z \longrightarrow \mathcal{F}$$

*is surjective. Then the set  $U_{Z, i \rightarrow i', \mathcal{F}}$  of points of  $Z$  where the composition*

$$\mathcal{E}_Z^i \longrightarrow \mathcal{E}_Z^{i'} \xrightarrow{\alpha_Z^{i'}} \mathcal{E}_Z \longrightarrow \mathcal{F}$$

*is surjective is open in  $Z$ .*

(2) *Let  $Y$  be another  $S$ -scheme. Then the set of  $S$ -morphisms  $g : Y \rightarrow Z$  such that  $g^*(\mathcal{F}) \in \gamma_i(Y)$  is the set of  $S$ -morphisms such that  $g(Y) \subset U_{Z, i \rightarrow i', \mathcal{F}}$ .*

A direct application of Lemma 1.3.9 concludes the proof of Lemma 1.3.8.

*Proof of Lemma 1.3.9:* (1). Follows immediately from Lemma 1.3.1(a).

(2). Note that  $g^*(q) : \mathcal{E}_Y \rightarrow g^*(\mathcal{F})$  is again a quotient homomorphism and that  $g^*(\alpha_Z^{i,i'}) = \alpha_Y^{i,i'}$ . By Lemma 1.3.1(c), the set of points  $y$  of  $Y$  where the localization of  $g^*(q) \circ \alpha_Y^{i,i'} \circ \alpha_Y^{i,i'} = g^*(q) \circ \alpha_Y^i$  is surjective is thus equal to  $g^{-1}(U_{Z, i, \mathcal{F}}) \subset Y$  (see the proof of Lemma 1.3.3 for the definition of  $U_{Z, i, \mathcal{F}}$ ). Now, since the functor  $\gamma_i$  is covered by functors of the form  $F_H$ , it follows that  $g^{-1}(U_{Z, i, \mathcal{F}})$  must coincide with  $Y$ .  $\square$   $\square$

**Proposition 1.3.10.** *Let  $\mathcal{E}$  be a quasi-coherent sheaf over the scheme  $S$ . Then, as  $i$  varies, the functors  $\gamma_i$  form an open covering of the functor  $\gamma$ . Furthermore, we have*

$$\mathrm{Grass}_n(\mathcal{E}) = \varinjlim_i \underline{G}(n, \mathcal{E})_i.$$

*Proof:* That each of the  $\gamma_i$ 's is an open subfunctor of  $\gamma$  was already established in Lemma 1.3.7. As in the previous Lemmas, for an  $S$ -scheme  $Z$  let  $U_{Z,i,\mathcal{F}}$  be the open subscheme of  $Z$  representing the usual fiber product functor

$$T \mapsto \gamma_i(T) \times_{\gamma(T)} \mathrm{Hom}_S(T, Z).$$

We show that the  $U_{Z,i,\mathcal{F}}$ 's cover  $Z$  as  $i$  varies. It is enough to show that the statement holds on points. Let then  $\mathcal{F} \in \gamma(Z)$ . At a closed point  $z \in Z$  we have in particular a locally free rank  $n$  sheaf  $\mathcal{F}_z$  generated by the localization at  $z$  of the  $n$  sections  $s_a$  (which were introduced on page 7) and an  $n$ -dimensional  $\kappa(z)$ -vector space  $\mathcal{F} \otimes_{\mathcal{O}_{Z,z}} \kappa(z)$  with basis the  $s_a(z)$ 's. This said, since  $\mathcal{F}$  is a quotient of finite type of the inductive limit  $\mathcal{E}$ , there must exist an index  $i$  and a surjection

$$\mathcal{E}^i \otimes_{\mathcal{O}_{Z,z}} \kappa(z) \rightarrow \mathcal{F} \otimes_{\mathcal{O}_{Z,z}} \kappa(z).$$

Thus, by Lemma 1.3.1(b), we obtain a surjection  $\mathcal{E}_{Z,z}^i \rightarrow \mathcal{F}_z$ , hence  $z \in U_{Z,i,\mathcal{F}}$  by definition. This concludes our argument.  $\square$

**Remark 1.3.11.** Note that Proposition 1.3.10 implies that our construction of the Grassmannian is independent of the filtration of the sheaf  $\mathcal{E}$  which we used.

## 2 Representability of the quasi-coherent *quot* functor.

Throughout this section  $S$  will be a noetherian scheme defined over some fixed algebraically closed field  $\mathbf{k}$ , and  $X$  will be a projective  $S$ -scheme of finite type. By a coherent sheaf on  $X$  we will mean a quasi-coherent  $\mathcal{O}_X$ -module of finite presentation.

### 2.1 Preliminaries

Let now  $T$  be another  $S$ -scheme, let  $\pi_X : X \times_S T \rightarrow X$  be the projection and denote by  $\mathcal{E}_T$  the pullback  $\pi_X^* \mathcal{E}$ , where  $\mathcal{E}$  is a quasi-coherent  $\mathcal{O}_X$ -module. For a numerical polynomial  $h \in \mathbb{Q}[t]$ , the *quot* functor

$$\eta_{\mathcal{E},h} := \mathrm{quot}_h^{\mathcal{E}} : (\mathbf{Sch}/S)^\circ \rightarrow \mathbf{Set},$$

is defined as

$$(2.1.1) \quad \eta_{\mathcal{E},h}(T) = \left\{ \mathcal{K} \subset \mathcal{E}_T \mid \begin{array}{l} \mathcal{E}_T/\mathcal{K} \text{ is coherent, flat over } \mathcal{O}_T, \\ \text{and has Hilbert polynomial } h \end{array} \right\},$$

together with pullback on morphisms.

Grothendieck's fundamental theorem reads as follows.

**Theorem 2.1.2** ([Gr]). *Let  $X$  be a projective  $S$ -scheme and let  $\mathcal{E}$  be a coherent sheaf on  $X$ . Then, the functor  $\eta_{\mathcal{E},h}$  is represented by a projective  $S$ -scheme  $\text{Quot}_h(\mathcal{E})$ . Moreover, there exists a coherent quotient  $\mathcal{Q} \in \eta_{\mathcal{E},h}(\text{Quot}_h(\mathcal{E}))$  such that, for any  $S$ -scheme  $T$ , the morphism of functors*

$$\text{Hom}_S(T, \text{Quot}_h(\mathcal{E})) \ni g \longmapsto g^* \mathcal{Q} \in \eta_{\mathcal{E},h}(T)$$

*is a natural isomorphism.*

On the other hand, without assuming coherence of  $\mathcal{E}$  we have

**Lemma 2.1.3.**  *$\eta_{\mathcal{E},h}$  is a sheaf in the Zariski topology.*

*Proof:* Let  $\{T_i \rightarrow T\}_i$  be a covering of the  $S$ -scheme  $T$  and let  $\mathcal{F}_i \in \eta_{\mathcal{E},h}(T_i)$ . In the usual notation for restrictions, suppose that  $\mathcal{F}_{i,j} = \mathcal{F}_{j,i} \in \eta_{\mathcal{E},h}(T_i \times_T T_j)$ , we want to find a unique sheaf  $\mathcal{F} \in \eta_{\mathcal{E},h}(T)$  whose restriction to  $T_i$  coincides with  $\mathcal{F}_i$ . The existence of such an object follows from the fact that our  $\eta_{\mathcal{E},h}$  is contained in the (homotopical) degree zero truncation of the stack  $\mathcal{Coh}_{X/S}$  (see [LMB]). Moreover  $\mathcal{F}$  has Hilbert polynomial  $h$  by semicontinuity, in particular by constance of the Hilbert polynomial on connected components.  $\square$

Given a quasi-coherent sheaf  $\mathcal{E}$  on  $X$ , our aim here is to construct an object  $\text{IQuot}_h^X(\mathcal{E})$ , possibly in the category of  $S$ -schemes, that represents the functor  $\eta_{\mathcal{E},h}$ .

Keeping the notation of Lemma 1.3.3, define the subfunctor

$$(2.1.4) \quad \eta_{\mathcal{E},h,i}(T) := \left\{ \mathcal{K} \in \eta_{\mathcal{E},h}(T) \mid \begin{array}{l} \text{the composite } \mathcal{E}_T^i \xrightarrow{\alpha_T^i} \mathcal{E}_T \xrightarrow{q} \mathcal{E}_T/\mathcal{K} \\ \text{is surjective} \end{array} \right\},$$

for an index  $i$ . Since  $\eta_{\mathcal{E},h,i}$  is a subfunctor of  $\eta_{\mathcal{E},h}$ , it is also a sheaf of sets.

## 2.2 Main Results

Let  $m \in \mathbb{Z}$ . Recall that a coherent sheaf  $\mathcal{G}$  on a polarized projective  $\text{Spec}(\mathbf{k})$ -scheme  $X$  is said to be  $m$ -regular (or of *Castelnuovo-Mumford regularity*  $m$ ) if

$$H^\alpha(X, \mathcal{G}(m - \alpha)) = 0,$$

for all  $\alpha > 0$ . Now, if

$$0 \rightarrow \mathcal{F}' \rightarrow \mathcal{F} \rightarrow \mathcal{F}'' \rightarrow 0$$

is a short exact sequence of coherent sheaves over  $X$ , additivity of the Euler characteristic on exact sequences implies that the regularity of  $\mathcal{F}$  is bounded by the maximum of the regularities of  $\mathcal{F}'$  and  $\mathcal{F}''$  (see e.g. [Mum, Lecture 14]).

The following Lemma is the main step in our construction.

**Lemma 2.2.1.** *Let  $\mathcal{E}$  be a quasi-coherent  $\mathcal{O}_X$ -module. Then for  $i \leq a \leq b$  we have an affine morphism*

$$Q_i^b \rightarrow Q_i^a,$$

*from the scheme representing the functor  $\eta_{\mathcal{E}^b, h, i}$  to the scheme representing  $\eta_{\mathcal{E}^a, h, i}$ . In particular, since  $Q_i^i := \text{Quot}_h(\mathcal{E}^i)$ , the morphism*

$$Q_i^a \rightarrow \text{Quot}_h(\mathcal{E}^i)$$

*is affine.*

*Proof:* We start by recalling the classical Grassmannian embedding of [Gr], then we will show that such embedding is compatible with our construction.

First, note that if  $\mathcal{E}^i$ ,  $\mathcal{E}^a$  and  $\mathcal{E}^b$  are three coherent sheaves on  $X$ , we can find a large enough integer  $m$  such that all three of them are  $m$ -regular. Next, recall that  $m$ -regularity of the coherent sheaf  $\mathcal{E}^i$  allows us to get, for any scheme  $T$  over  $S = \text{Spec}(\mathbf{k})$  and for any  $T$ -flat quotient homomorphism  $\mathcal{E}_T^i \rightarrow \mathcal{F}$  with kernel  $\mathcal{K}$ , a short exact sequence of locally free sheaves over  $T$

$$0 \rightarrow \pi_{T*}\mathcal{K}(m) \rightarrow H^0(X, \mathcal{E}^i(m)) \otimes_{\mathbf{k}} \mathcal{O}_T \rightarrow \pi_{T*}\mathcal{F}(m) \rightarrow 0.$$

We thus obtain an embedding of the functor  $\eta_{\mathcal{E}^i, h}$  into the functor  $\gamma_{\mathcal{E}^i, h(m)}$ . This in turn allows us to find a flattening stratum of the Grassmannian (in

the sense of [Mum]) that represents the functor  $\eta_{\mathcal{E}^i, h}$ . More precisely, in the current hypotheses the construction of the image of the Grassmannian embedding that we have to use is the one at the beginning of [CFK]. Next, consider the homomorphism  $\mathcal{E}^i \rightarrow \mathcal{E}^a$ . This gives a natural transformation

$$\eta_{\mathcal{E}^a, h, i} \Rightarrow \eta_{\mathcal{E}^i, h},$$

defined in the obvious way. The above transformation yields in turn a morphism of schemes

$$Q_i^a \rightarrow Q_i^i,$$

by representability of the *quot* functor of a coherent sheaf and Yoneda's lemma. We claim that such a morphism is affine. We will use the covering induced on  $\text{Quot}_h(\mathcal{E}^i)$  by the one of the Grassmannian which we constructed in Section 1.

In fact, from  $\mathcal{E}^i \rightarrow \mathcal{E}^a$  we get a commutative square

$$\begin{array}{ccc} Q_i^a & \longrightarrow & Q_i^i \\ \downarrow & & \downarrow \\ G_i^a & \longrightarrow & G_i^i, \end{array}$$

where  $G_i^a$  is the open part of the Grassmannian  $\text{Grass}_{h(m)}(H^0(X, \mathcal{E}^a(m)))$  whose points are isomorphism classes of quotients of  $H^0(X, \mathcal{E}^i(m))$  and the vertical arrows denote the respective Grassmannian embeddings. By Lemma 1.3.3 the lower arrow is an affine morphism, so we can conclude that  $Q_i^a \rightarrow Q_i^i$  is also an affine morphism by restricting the lower arrow to the respective flattening strata.

More generally, from the homomorphism  $\mathcal{E}^a \rightarrow \mathcal{E}^b$ , we get a natural transformation

$$\eta_{\mathcal{E}^b, h, i} \Rightarrow \eta_{\mathcal{E}^a, h, i},$$

and a resulting morphism of schemes  $Q_i^b \rightarrow Q_i^a$ . Keeping the notation as above we have another commutative diagram

$$\begin{array}{ccc} Q_i^b & \longrightarrow & Q_i^a \\ \downarrow & & \downarrow \\ G_i^b & \longrightarrow & G_i^a, \end{array}$$

which allows us to conclude that  $Q_i^b \rightarrow Q_i^a$  is affine, as well, by essentially the same argument.



Finally, let  $S$  be any noetherian scheme over  $\mathbf{k}$ . Then the statement follows from what we proved above plus the base change property of affine morphisms.  $\square$

**Lemma 2.2.2.** *Let  $\mathcal{E}$  be a quasi-coherent  $\mathcal{O}_X$ -module. Then the functor  $\eta_{\mathcal{E},h,i}$  is represented by*

$$\mathcal{Q}(h, \mathcal{E})_i := \varprojlim_{a \geq i} Q_i^a,$$

*which is a quasi-compact scheme over  $S$ .*

*Proof:* From Lemma 2.2.1 we see that all of the morphisms in the filtering projective diagram

$$(2.2.3) \quad ((Q_i^a)_{a \geq i}, Q_i^a \leftarrow Q_i^b)$$

are affine. As in the proof of Lemma 1.3.5, we then use the fact that the projective limit of a system of quasi-compact schemes and affine morphisms is again a quasi-compact scheme.

In order to conclude, it remains to prove that the scheme  $\mathcal{Q}(h, \mathcal{E})_i$  obtained as the projective limit of the diagram (2.2.3) represents the functor  $\eta_{\mathcal{E},h,i}$ . For this, the argument we used in Lemma 1.3.5 for the functors  $\gamma_i$  and the schemes  $\mathcal{G}(n, \mathcal{E})_i$  applies mutatis mutandis.  $\square$

**Lemma 2.2.4.** *For  $i \leq j$  we have an open embedding of schemes*

$$Q_i^a \rightarrow Q_j^a.$$

*Proof:* As usual, we prove the corresponding statement at the level of functors, i.e., we show that for every  $S$ -scheme  $Z$  the fiber product functor

$$T \mapsto \eta_{\mathcal{E}^a,h,i}(T) \times_{\eta_{\mathcal{E}^a,h,j}(T)} \mathrm{Hom}_S(T, Z),$$

is represented by an open subscheme of  $Z$ . Now, by definition of  $\eta_{\mathcal{E}^a,h,i}$  we have a surjective composition

$$\mathcal{E}_T^i \rightarrow \mathcal{E}_T^j \rightarrow \mathcal{E}_T^a \rightarrow \mathcal{F},$$

where the last homomorphism is the canonical quotient. Therefore the claim follows from Lemma 1.3.9 as did the analogous statement for the Grassmannian in Section 1.  $\square$

Now, note that we have a commutative ladder

$$\begin{array}{ccc}
\vdots & \longrightarrow & \vdots \\
\downarrow & & \downarrow \\
Q_i^b & \longrightarrow & Q_j^b \\
\downarrow & & \downarrow \\
Q_i^a & \longrightarrow & Q_j^a \\
\downarrow & & \downarrow \\
\vdots & \longrightarrow & \vdots
\end{array}$$

where the vertical arrows are surjections and the horizontal ones are open morphisms by the last Lemma. This allows us to take the inductive limit over the lower indices and get a morphism of schemes

$$(2.2.5) \quad \varprojlim Q(h, \mathcal{E})_i \rightarrow \varprojlim Q(h, \mathcal{E})_j.$$

Finally, define

$$(2.2.6) \quad \mathrm{IQuot}_h(\mathcal{E}) := \varinjlim_i \varprojlim Q(h, \mathcal{E})_i,$$

where  $\left(\varprojlim Q(h, \mathcal{E})_i\right)_i$  is the system of quasi-compact schemes and embeddings of the form (2.2.5) resulting from Lemma 2.2.2 and Lemma 2.2.4.

**Theorem 2.2.7.** *In the above notation, the functor  $\eta_{\mathcal{E},h}$  is covered by the functors  $\eta_{\mathcal{E},h,i}$ .*

*Proof:* It remains to show that the subfunctors  $\eta_{\mathcal{E},h,i}$  cover  $\eta_{\mathcal{E},h}$  as  $i$  varies. As in the case of the Grassmannian, it is enough to check this pointwise. Let  $\mathcal{F} \in \eta_{\mathcal{E},h}(Z)$ ,  $z \in Z$ , and consider the  $\kappa(z)$ -module of finite type  $\mathcal{F} \otimes_{\mathcal{O}_{Z,z}} \kappa(z)$ . Then there is an index  $i$  such that we have a surjection

$$\mathcal{E}^i \otimes_{\mathcal{O}_{Z,z}} \kappa(z) \rightarrow \mathcal{F} \otimes_{\mathcal{O}_{Z,z}} \kappa(z).$$

At this point, the fact that there is a surjection  $\mathcal{E}_{Z,z}^i \rightarrow \mathcal{F}_z$  follows from flatness of the quotient  $\mathcal{F}$  and Nakayama's Lemma.  $\square$

At this stage, using the same tools that we used in Section 1, we can prove in essentially the same way that the morphisms (2.2.5) are actually open embeddings and that the functors  $\eta_{\mathcal{E},h,i}$  are open subfunctors of  $\eta_{\mathcal{E},h}$ . Therefore we can improve upon the statement of Lemma 2.2.7 as follows.

**Theorem 2.2.8.** *The functor  $\eta_{\mathcal{E},h}$  is represented by the scheme  $\mathrm{IQuot}_h(\mathcal{E})$ .*

*Proof:* As we said right before the statement, the functors  $\eta_{\mathcal{E},h,i}$  are open subfunctors of  $\eta_{\mathcal{E},h}$ . Moreover, by Lemma 2.2.2 such functors are representable and by Lemma 2.2.7 they form a covering of the functor  $\eta_{\mathcal{E},h}$ . All of the above plus Lemma 2.1.3 allow us to conclude.  $\square$

**Example 2.2.9.** Let  $k$  be a field and let  $X = \mathrm{Spec} k$  be a point. An object in  $\mathcal{F} \in \mathrm{QCoh}(X)$  is then a (possibly infinite dimensional) vector space over  $k$ , say  $\mathcal{F} = V$ .

In this case  $\mathrm{IQuot}_h^X(\mathcal{F})$  reduces to a schematic Grassmannian of quotients of a certain dimension prescribed by the Hilbert polynomial.

### 2.3 Uniformly regular sheaves and a “large scale” Grassmannian embedding

Let again  $\mathcal{E}$  be a quasi-coherent  $\mathcal{O}_X$ -module, and let  $S = \mathrm{Spec}(\mathbf{k})$ . We will show that in this case it is possible to prove an analog of the classical Grassmannian embedding.

Motivated by the discussion preceding Lemma 2.2.1, we make the following definition.

**Definition 2.3.1.** A quasi-coherent sheaf over a projective  $\mathbf{k}$ -scheme  $X$  will be said to be *uniformly  $m$ -regular* if there is an integer  $m$  such that the Castelnuovo-Mumford regularities of its coherent approximations in the sense of Proposition 1.1.1 and Remark 1.1.2(2) are all less or equal to  $m$ .

**Lemma 2.3.2.** *Let  $\mathcal{E}$  be a uniformly  $m$ -regular quasi-coherent  $\mathcal{O}_X$ -module. Then there is a closed embedding*

$$(2.3.3) \quad \varprojlim_i \mathcal{Q}(h, \mathcal{E})_i \rightarrow \varprojlim_i \mathcal{G}(h(m), \varinjlim_j H^0(X, \mathcal{E}^j(m)))_i.$$

*Proof:* We go back to considering the components of the source and target schemes considered as projective limits. In our usual notation, we have a commutative ladder diagram

$$\begin{array}{ccc}
\vdots & \longrightarrow & \vdots \\
\downarrow & & \downarrow \\
Q_i^b & \longrightarrow & G_i^b \\
\downarrow & & \downarrow \\
Q_i^a & \longrightarrow & G_i^a \\
\downarrow & & \downarrow \\
\vdots & \longrightarrow & \vdots
\end{array}$$

where the vertical arrows are surjective affine morphisms and the horizontal ones are the restrictions of the respective Grassmannian embeddings. The vertical morphisms being affine, we can reduce to proving the statement locally.

Let then  $A$  and  $B$  be two rings such that

$$A = \varinjlim_{\beta} A^{\beta}, \quad B = \varinjlim_{\beta} B^{\beta},$$

and suppose  $A^{\beta} \xleftarrow{\psi_{\beta}} B^{\beta}$  is a surjective (quotient) homomorphism for all  $\beta$ , i.e.,  $A^{\beta} = B^{\beta}/I^{\beta}$  where  $I^{\beta} = \ker(\psi_{\beta})$ . We can then realize  $A$  as a global quotient of  $B$  modulo the ideal

$$\varinjlim_{\beta} I^{\beta}.$$

This allows us to establish the statement.  $\square$

Writing  $\underline{Q}(i) := \varprojlim \underline{Q}(h, \mathcal{E})_i$  and  $\underline{G}(i) := \varprojlim \underline{G}(h(m), \varinjlim_j H^0(X, \mathcal{E}^j(m)))_i$  to simplify the notation, we thus have another commutative ladder diagram

$$\begin{array}{ccc}
\vdots & \longrightarrow & \vdots \\
\downarrow & & \downarrow \\
\varprojlim Q(i) & \longrightarrow & \varprojlim G(i) \\
\downarrow & & \downarrow \\
\varprojlim Q(i') & \longrightarrow & \varprojlim G(i') \\
\downarrow & & \downarrow \\
\vdots & \longrightarrow & \vdots
\end{array}$$

where the horizontal arrows are the closed embeddings resulting from Lemma 2.3.2 and the vertical ones are the open embeddings (2.2.5) from the previous subsection. In analogy with the concept of quasi-projectivity in finite dimensions, we make the following definition.

**Definition 2.3.4.** We call *quasi-closed* an embedding of ind-schemes resulting from a limit of a ladder like the above one.

Our final result for this section is

**Proposition 2.3.5.** *Let  $\mathcal{E}$  be a uniformly  $m$ -regular quasi-coherent sheaf on a  $\mathbf{k}$ -projective scheme  $X$ . Then there is a quasi-closed embedding*

$$\mathrm{Quot}_h^X(\mathcal{E}) \hookrightarrow \mathrm{Grass}_{h(m)} \left( \varinjlim_j H^0(X, \mathcal{E}^j(m)) \right).$$

*Proof:* Follows at once from the argument preceding the statement.  $\square$

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